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2006 J. Phys. A: Math. Gen. 39 2143

(http://iopscience.iop.org/0305-4470/39/9/010)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 2143–2149 doi:10.1088/0305-4470/39/9/010

One-dimensional Coulomb-like problem in deformed space with minimal length

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Received 4 August 2005, in final form 17 January 2006
Published 15 February 2006
Online at stacks.iop.org/JPhysA/39/2143

Abstract

The spectrum and eigenfunctions in the momentum representation for a 1D Coulomb-like potential with deformed Heisenberg algebra leading to minimal length are found exactly. It is shown that the correction due to the deformation is proportional to the square root of the deformation parameter. We obtain the same spectrum using the Bohr–Sommerfeld quantization condition.

PACS numbers: 03.65.Ge, 02.40.Gh

1. Introduction

Quantum mechanics with modification of the usual canonical commutation relations has attracted a lot of attention recently. Such works are motivated by several independent lines of investigation in string theory and quantum gravity, which suggest the existence of a finite lower bound to the possible resolution of length ΔX [1–3].

In this paper we consider 1D quantum mechanics with the following deformation [4–6]:

$$[X, P] = \mathrm{i}\hbar(1 + \beta P^2). \tag{1}$$

Here βP^2 is a small correction. If $\beta = 0$ we obtain the usual algebra. Such a deformation implies that there exists a minimal resolution length $\Delta X \geqslant \hbar\sqrt{\beta}$ [5], i.e., there is no possibility of measuring coordinate X with accuracy more than ΔX. Note that the deformed commutation relation (1) gives the same uncertainty relation as was suggested in string theory [1]. That is why it was assumed that the physical position and momentum operators could be identified with X and P operators satisfying the deformed commutation relation (1). Thus, we demand that these operators as well the Hamiltonian H be Hermitian operators.

The use of the deformed commutation relation (1) brings new difficulties in solving the quantum problems. As far as we know there are only a few problems for which spectra have been found exactly. They are the one-dimensional oscillator [5], D-dimensional isotropic harmonic oscillator [7] and three-dimensional relativistic Dirac oscillator [8]. Note that in the one-dimensional case the harmonic oscillator problem has been solved exactly [9, 10] for more general deformation leading to nonzero uncertainties in both position and momentum.

In this paper, we solve the following eigenvalue problem:

$$P^2\psi - \frac{\alpha}{X}\psi = E\psi. \tag{2}$$

Here we put $\hbar = 1$ and $2m = 1$, $1/X$ means inverse operator of operator X. Recently, in the undeformed case this problem was considered in [11, 12]. There exists a similar singular potential $-\alpha/|X|$ which had been considered in detail. Since these potentials are singular there exist a variety of approaches to the quantization conditions (paper [13] can be considered as a mini review on this topic) and due to different symmetric extensions of operator $-\alpha/|X|$ different spectra are obtained [14].

Although the potential $-\alpha/X$ has not been studied so intensively as $-1/|X|$, it has some interesting application in theoretical physics. In [15], it is shown that this potential appears in the investigation of mass spectra of mesons (quark–antiquark systems) in the framework of Dirac oscillators. The problem, in the centre-of-mass frame, was reduced to a familiar radial equation but with the singular potential $V = -b^2/(r^2 - a^2) \sim -b^2/2ax$ where $x = r - a$. The important part of this potential is just one-dimensional Coulomb-like problem which was studied in [15]. At the scale of energies and lengths of this system one can expect the appearance of measurable influence of the minimal length on the energy spectrum. This is one of the main physical motivations of our studies of the potential $-\alpha/X$ in deformed space with minimal length.

Also, the potential $-\alpha/X$ may have application in the physics of semiconductors and insulators [11]. The possibility of using deformed commutation relation (1) in condensed matter physics for the description of nonpointlike quasi-particles was pointed out by Kempf [16]. In this case, the minimal length is interpreted as a free parameter linked with the structure of nonpointlike particles and their finite size; and no attempt is made to give an explicit link with some fundamental properties of the particles [17]. We also would like to present a toy model which is described by this potential. In this model, a point dipole with constant orientation (constant orientation of the dipole can be provided by a strong uniform electric field) is moving along a line in the field of a uniformly charged wire which is perpendicular to the motion line.

In the deformed case, the spectrum of the three-dimensional hydrogen atom was approximately found by Brau with the help of perturbation theory [18], numerically by Benczik and colleagues [19]. A comparison between the 'space curvature' effects and minimal length effects was made in [20]. The authors of [21] claimed that they found the spectrum, but it seems that there exists some incorrectness in their paper, which we discuss below.

The consideration of simple quantum systems in the deformed space gives the possibility of studying the influence of deformation on energy spectra and finding the correction caused by deformation. Comparing this correction with the experimental data it is possible to estimate the value of deformation parameter (see, for instance, [18]). Thus, the investigation of quantum-mechanical problems in the deformed space is interesting from the mathematical as well as from the physical points of view.

This paper is organized as follows. In section 2, we define X and P operators and find solutions of the eigenvalue equation (2). In section 3, we define action of $1/X$ operator on eigenfunctions and discuss the quantization condition. In section 4, we obtain spectrum from the Bohr–Sommerfeld quantization condition. And finally, in section 5, we discuss our results.

2. Momentum representation

There are different representations of algebra (1). One of them is the so-called quasi-coordinate representation [5] $X = x$ and $P = \frac{1}{\sqrt{\beta}} \tan \sqrt{\beta} p$, where $[x, p] = \mathrm{i}$. It is called quasi-coordinate representation because X does not possess eigenfunctions for which the mean value of kinetic energy is finite and therefore eigenstates of functions of operator X do not belong to the physical states (for details see [5]).

Therefore, we prefer to use the momentum representation. In this representation the momentum and coordinate operators read

$$P = p, \qquad X = \mathrm{i}(1 + \beta p^2) \frac{\mathrm{d}}{\mathrm{d}p}. \tag{3}$$

Operator X defined in such a way is a Hermitian operator with the following definition of scalar product [5]:

$$\langle \phi | \psi \rangle = \int_{-\infty}^{\infty} \frac{\phi^*(p)\psi(p)}{1 + \beta p^2} \, \mathrm{d}p. \tag{4}$$

For the undeformed case, the action of inverse operator $1/X$ has been expressed in the following way [22]:

$$\frac{1}{X}\psi(p) = -\mathrm{i} \int_{-\infty}^{p} \psi(q) \, \mathrm{d}q. \tag{5}$$

In the deformed case we can express it in a similar way,

$$\frac{1}{X}\psi(p) = -\mathrm{i} \int_{-\infty}^{p} \frac{\psi(q)}{1 + \beta q^2} \, \mathrm{d}q. \tag{6}$$

For such a definition $\frac{1}{X} X \psi(p) = X \frac{1}{X} \psi(p) = \psi(p)$. But in the undeformed case the application formula (5) leads to the existence of the only trivial solution $\psi(p) = 0$ [12]. Using the same procedure as in [12] for the deformed case we obtain the same trivial solution.

In order to obtain non-trivial solutions it is necessary to redefine $1/X$ operator slightly. We rewrite formula (6) as

$$\frac{1}{X}\psi(p) = -\mathrm{i} \int_{-\infty}^{p} \frac{\psi(q)}{1 + \beta q^2} \, \mathrm{d}q + c, \tag{7}$$

where c is a constant. For such a definition $X \frac{1}{X} = 1$, but $\frac{1}{X} X \neq 1$. We shall find the value of this constant below. Note that in the undeformed case existence of c in the momentum representation corresponds to derivative discontinuity of eigenfunction at the origin in the coordinate representation [12].

Multiplying the eigenvalue equation (2) by X we obtain a new equation which does not depend on constant c:

$$X P^2 \psi - \alpha \psi = E X \psi. \tag{8}$$

Its explicit form reads

$$\mathrm{i}(1 + \beta p^2) \left[p^2 \psi'(p) + 2p\psi(p) - E\psi'(p) \right] - \alpha \psi(p) = 0, \tag{9}$$

where $'$ denotes the derivative with respect to p. The solution of the last equation is

$$\psi_\epsilon(p) = \frac{C_\epsilon}{\epsilon + p^2} \exp\left[\frac{-\mathrm{i}\alpha}{1 - \epsilon\beta} \left(\frac{1}{\sqrt{\epsilon}} \arctan \frac{p}{\sqrt{\epsilon}} - \sqrt{\beta} \arctan \sqrt{\beta} p \right) \right], \tag{10}$$

where $\epsilon = -E$ and a normalization constant C_ϵ reads

$$C_\epsilon = \sqrt{\frac{2}{\pi}} \epsilon^{\frac{3}{4}} \frac{1 + \sqrt{\epsilon\beta}}{\sqrt{1 + 2\sqrt{\epsilon\beta}}}.$$

If $\epsilon \leqslant 0$, then normalization integral for eigenfunction (10) $\langle\psi_\epsilon|\psi_\epsilon\rangle$ diverges. So, we require that $\epsilon > 0$.

We can rewrite this eigenfunction in the following way:

$$\psi_\epsilon(p) = \frac{C_\epsilon}{\epsilon + p^2}\left(\frac{\sqrt{\epsilon}+\mathrm{i}p}{\sqrt{\epsilon}-\mathrm{i}p}\right)^{-\frac{\alpha}{2\sqrt{\epsilon}(1-\epsilon\beta)}}\left(\frac{1+\mathrm{i}\sqrt{\beta}p}{1-\mathrm{i}\sqrt{\beta}p}\right)^{\frac{\alpha\sqrt{\beta}}{2(1-\epsilon\beta)}}. \tag{11}$$

In limit $\beta \to 0$ this eigenfunction is equivalent to the corresponding eigenfunction for the undeformed case from paper [22].

Function (10) satisfies equation (8) but it does not satisfy the initial equation (2) with operator $1/X$ in the form defined by (6). If we define operator $1/X$ in form (7) we can find such a constant c that eigenfunction (10) satisfies the eigenvalue equation (2).

The procedure presented in [21] leads to correct expressions of eigenfunctions in the one-dimensional case and eigenfunctions expression from that paper and expression (11) coincide. In [21], the quantization condition was chosen from the requirement of single-valuedness of eigenfunction (11). It was stated that this condition is equivalent to $\frac{\alpha}{2\sqrt{\epsilon}(1-\epsilon\beta)} = n$, but the influence of the term in the second parentheses on the single-valuedness was neglected. In the next section, we discuss the correct quantization condition.

3. Spectrum

To find constant c let us make some manipulations with equation (9). After dividing it by $\mathrm{i}(1+\beta p^2)$ and subsequent integration over p we obtain

$$p^2\psi(p) + \mathrm{i}\alpha\int_{-\infty}^{p}\frac{\psi(q)\,\mathrm{d}q}{1+\beta q^2} - \alpha c[\psi] = E\psi(p), \tag{12}$$

where $c[\psi]$ is an integration constant (with respect to p) being, in general, a functional of ψ.

The last equation (12) has the form of the eigenvalue equation (2). So, we can express the action of operator $1/X$ on an eigenfunction as follows:

$$\frac{1}{X}\psi(p) = -\mathrm{i}\int_{-\infty}^{p}\frac{\psi(q)\,\mathrm{d}q}{1+\beta q^2} + c[\psi]. \tag{13}$$

Substituting the expression for eigenfunctions (10) into the eigenvalue equation (12) we obtain

$$c[\psi_\epsilon] = \frac{1}{\alpha}\lim_{p\to-\infty}(p^2+\epsilon)\psi_\epsilon(p) = \frac{C_\epsilon}{\alpha}\exp\left(\frac{\mathrm{i}\alpha\pi}{2(\sqrt{\epsilon}+\sqrt{\beta}\epsilon)}\right). \tag{14}$$

We require that the Hamiltonian corresponding to the eigenvalue equation (2) be a Hermitian operator on its eigenfunction (10). It is obvious that operator p^2 is an Hermitian operator on these eigenfunctions. Thus, we require that operator $1/X$ be a Hermitian operator on the set of eigenfunctions

$$\left\langle\frac{1}{X}\psi_{\epsilon_i}\middle|\psi_{\epsilon_j}\right\rangle = \left\langle\psi_{\epsilon_i}\middle|\frac{1}{X}\,\psi_{\epsilon_j}\right\rangle. \tag{15}$$

Using expression (13) for operator $1/X$ we can rewrite this condition

$$\begin{aligned}\mathrm{i}\int_{-\infty}^{\infty}\frac{\psi_{\epsilon_j}(p)}{1+\beta p^2}\,\mathrm{d}p\int_{-\infty}^{p}\frac{\psi^*_{\epsilon_i}(q)}{1+\beta q^2}\,\mathrm{d}q + c^*\big[\psi_{\epsilon_i}\big]\int_{-\infty}^{\infty}\frac{\psi_{\epsilon_j}(p)}{1+\beta p^2}\,\mathrm{d}p \\ = -\mathrm{i}\int_{-\infty}^{\infty}\frac{\psi^*_{\epsilon_i}(p)}{1+\beta p^2}\,\mathrm{d}p\int_{-\infty}^{p}\frac{\psi_{\epsilon_j}(q)}{1+\beta q^2}\,\mathrm{d}q + c\big[\psi_{\epsilon_j}\big]\int_{-\infty}^{\infty}\frac{\psi^*_{\epsilon_i}(p)}{1+\beta p^2}\,\mathrm{d}p.\end{aligned} \tag{16}$$

According to the facts that

$$\int_{-\infty}^{\infty}f(p)\,\mathrm{d}p\int_{-\infty}^{p}g(q)\,\mathrm{d}q = \int_{-\infty}^{\infty}g(p)\,\mathrm{d}p\left[\int_{-\infty}^{\infty}f(q)\,\mathrm{d}q - \int_{-\infty}^{p}f(q)\,\mathrm{d}q\right],$$

and

$$\int_{-\infty}^{\infty} \frac{\psi_\epsilon(p)\,\mathrm{d}p}{1+\beta p^2} = \frac{2C_\epsilon}{\alpha} \sin g(\epsilon), \tag{17}$$

where

$$g(\epsilon) = \frac{\alpha\pi}{2(\sqrt{\epsilon}+\sqrt{\beta}\epsilon)},$$

we simplify condition (16) to

$$\sin[g(\epsilon_i) - g(\epsilon_j)] = 0.$$

As a result we have $g(\epsilon_i) - g(\epsilon_j) = \pi m$, where m is an integer. So, by declaring that some ϵ_0 belongs to the spectrum we put the following condition on the remaining eigenvalues ϵ:

$$\frac{\alpha}{2(\sqrt{\epsilon}+\sqrt{\beta}\epsilon)} = \frac{\alpha}{2(\sqrt{\epsilon_0}+\sqrt{\beta}\epsilon_0)} + m \tag{18}$$

or

$$\frac{\alpha}{2(\sqrt{\epsilon}+\sqrt{\beta}\epsilon)} = \delta + n, \tag{19}$$

where $n = m + \left[\frac{\alpha}{2(\sqrt{\epsilon_0}+\sqrt{\beta}\epsilon_0)}\right]$ is an integer (here $[x]$ denotes the integer part of x), parameter δ is a fractional part of $\frac{\alpha}{2(\sqrt{\epsilon_0}+\sqrt{\beta}\epsilon_0)}$ and is in the range $0 \leqslant \delta < 1$.

In fact, we obtain a family of spectra, each of them is characterized by a value of δ. The value of parameter δ can be calculated from the results of an experiment.

For the undeformed case it is supposed in many papers that $\delta = 0$; this corresponds to the eigenfunction vanishing at the origin $(\psi(x)\big|_{x\to 0} = 0)$. Let us analyse the case of $\delta = 0$ in the deformed case more thoroughly. We have

$$\frac{\alpha}{2(\sqrt{\epsilon}+\sqrt{\beta}\epsilon)} = n, \tag{20}$$

where n is an integer. It has no finite solutions for $n = 0$ and it is a quadratic equation in $\sqrt{\epsilon}$ if $n \neq 0$. Its two real solutions read as

$$\sqrt{\epsilon}_{1,2} = \frac{-1}{2\sqrt{\beta}} \pm \frac{1}{2\sqrt{\beta}}\sqrt{1+\frac{2\alpha}{n}\sqrt{\beta}}. \tag{21}$$

According to the fact that $\sqrt{\epsilon} > 0$ we have to keep only one solution with '+' and, moreover, we have to require that n and α have the same sign, for $\alpha > 0$ we obtain that n is a positive integer.

So, the spectrum of Hamiltonian (2) for $\delta = 0$ is expressed as follows:

$$E_n = -\epsilon = -\frac{1}{4\beta}\left(1-\sqrt{1+\frac{2\alpha}{n}\sqrt{\beta}}\right)^2, \qquad n = 1, 2, \ldots. \tag{22}$$

For small β energy spectrum can be approximated as

$$E_n = -\frac{\alpha^2}{4n^2} + \frac{\alpha^3}{4n^3}\sqrt{\beta} - \frac{5\alpha^4}{16n^4}\beta + o(\beta^{3/2}), \qquad n = 1, 2, \ldots.$$

In the limit $\beta \to 0$, the spectrum (22) coincides with the hydrogen atom spectrum with $\alpha = e^2, m = 1/2, \hbar = 1$.

It is interesting to note that if $\delta = 0$ then eigenfunction (10) is a single-valued function of complex variable p except two finite cuts (the first cut goes from $\mathrm{i}\sqrt{\epsilon}$ to $\mathrm{i}/\sqrt{\beta}$, the second one goes from $-\mathrm{i}\sqrt{\epsilon}$ to $-\mathrm{i}/\sqrt{\beta}$). The condition of single-valuedness is widely used as a quantization criterion [12, 21, 22], but in our opinion it is an unconvincing criterion.

In general, $\delta \neq 0$ and the energy spectrum can be obtained in a similar way and it reads

$$E_n = -\frac{1}{4\beta}\left(1 - \sqrt{1 + \frac{2\alpha}{n+\delta}\sqrt{\beta}}\right)^2, \qquad n = 0, 1, 2, \ldots. \tag{23}$$

Here, due to the fact that $\delta > 0$ we obtain one additional level with quantum number $n = 0$. If one tends δ to zero for $n = 0$ we obtain infinite negative energy and the corresponding eigenfunction vanishes everywhere. Similar properties appear in the undeformed case [13].

4. Semiclassical approach

For the 1D undeformed case, $[x, p] = \mathrm{i}$ and the Bohr–Sommerfeld quantization condition reads

$$2\pi(n+\delta) = \oint p\,\mathrm{d}x, \tag{24}$$

where n is an integer, and δ is a parameter which depends on the boundary conditions [23] $(0 \leqslant \delta < 1)$. In this section, we construct the Bohr–Sommerfeld quantization condition for the deformed case.

Operator $X = \mathrm{i}(1+\beta p^2)\frac{\mathrm{d}}{\mathrm{d}p}$ we rewrite as

$$X = (1+\beta p^2)x, \qquad x = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p}. \tag{25}$$

Here x, p are canonical variables which satisfy $[x, p] = \mathrm{i}$.

Classical Hamiltonian corresponding to the system reads

$$H(x, p) = p^2 - \frac{\alpha}{(1+\beta p^2)x}. \tag{26}$$

From the energy conservation law $H(x, p) = E$ we can express p as a function of x but it is cumbersome to integrate the respective function. Instead, we use the identity $\oint p\,\mathrm{d}x = -\oint x\,\mathrm{d}p$ and express x as a function of p:

$$x = \frac{\alpha}{(1+\beta p^2)(p^2 - E)}. \tag{27}$$

When the particle moves from the origin to some turning point momentum p changes from $+\infty$ to 0, when the particle returns to the origin, p changes from 0 to $-\infty$. Therefore, $-\oint x\,\mathrm{d}p = \int_{-\infty}^{\infty} x\,\mathrm{d}p$ and

$$2\pi(n+\delta) = \int_{-\infty}^{\infty} \frac{\alpha}{(1+\beta p^2)(p^2 - E)}\,\mathrm{d}p = \frac{\pi\alpha}{\sqrt{\epsilon} + \epsilon\sqrt{\beta}}. \tag{28}$$

Here we take into account that for bound states energy $E = -\epsilon$ is negative.

Condition (28) coincides with condition (19) and we recover the same spectrum as in formula (23).

5. Discussions

In this section, we discuss two interesting properties of the problem considered in this paper and the Bohr–Sommerfeld quantization condition.

The existence of different spectral families means that there exist different extensions of operator $1/X$. We think that from the physical point of view it signifies that we can approximate a singular potential with different regular ones and as a result we obtain different spectra (cf [13]). As one can see, the existence of minimal length does not remove the singularity of potential $1/X$.

Another interesting property is that the first correction to the energy spectrum is proportional to $\sqrt{\beta}$. It brings an opportunity to reveal the existence of the deformed commutation relation (1) for smaller β. For previously solved problems (harmonic oscillator [16], hydrogen atom [18]) the correction is proportional to β. On the other hand, it is difficult to say if it is possible, using the present experimental setup, to find corrections to energy spectrum caused by deformation of space with minimal length in quantum systems described by 1D potential $1/X$.

We also derive the energy spectrum using the Bohr–Sommerfeld quantization condition. It is interesting to note that semiclassical result coincides with the exact result obtained from the Schrödinger equation.

Acknowledgments

We thank Dr Andrij Rovenchak and Yuri Krynytskyi for helpful discussions. We also would like to thank Dr Andrij Rovenchak for a careful reading of the paper.

References

[1] Gross D J and Mende P F 1988 *Nucl. Phys.* B **303** 407
[2] Maggiore M 1993 *Phys. Lett.* B **304** 65
[3] Witten E 1996 *Phys. Today* **49** 24
[4] Kempf A 1994 *J. Math. Phys.* **35** 4483
[5] Kempf A, Mangano G and Mann R B 1995 *Phys. Rev.* D **52** 1108
[6] Kempf A and Mangano G 1997 *Phys. Rev.* D **55** 7909
[7] Chang L N, Minic D, Okamura N and Takeuchi T 2002 *Phys. Rev.* D **65** 125027
[8] Quesne C and Tkachuk V M 2005 *J. Phys. A: Math. Gen.* **38** 1743
[9] Quesne C and Tkachuk V M 2003 *J. Phys. A: Math. Gen.* **36** 10373
[10] Quesne C and Tkachuk V M 2004 *J. Phys. A: Math. Gen.* **37** 10095
[11] Reyes J A and del Castillo-Mussot M 1999 *J. Phys. A: Math. Gen.* **32** 2017
[12] Ran Y, Xue L, Hu S and Su R-K 2000 *J. Phys. A: Math. Gen.* **33** 9265
[13] Gordeyev A N and Chhajlany S C 1997 *J. Phys. A: Math. Gen.* **30** 6893
[14] Tsutsui I, Fulop T and Cheon T 2003 *J. Phys. A: Math. Gen.* **36** 275
[15] Moshinsky M 1993 *J. Phys. A: Math. Gen.* **26** 2445
[16] Kempf A 1997 *J. Phys. A: Math. Gen.* **30** 2093
[17] Brau F 2003 *J. Phys. A: Math. Gen.* **36** 1523 and references therein
[18] Brau F 1999 *J. Phys. A: Math. Gen.* **32** 7691
[19] Benczik S, Chang L N, Minic D and Takeuchi T 2005 *Phys. Rev.* A **72** 012104
[20] Nieto L M, Rosu H C and Santander M 1999 *Mod. Phys. Lett.* A **14** 2463
[21] Akhoury R and Yao Y-P 2003 *Phys. Lett.* B **572** 37
[22] Nunez Yepez H N, Vargas C A and Brito A L S 1987 *Eur. J. Phys.* **8** 189
[23] Landau L D and Lifshits E M 1977 *Quantum Mechanics: Non-relativistic Theory* (New York: Pergamon)